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TECHNICAL REPORT

A METHOD FOR THE NUMERICAL SOLUTION OF A NONLINEAR DIFFERENTIAL EQUATION RESULTING FROM AN EXTENSION OF NEETESON'S EQUATIONS ON FERRITE CORES

by
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SUMMARY

Some differences between the mathematical treatment of magnetic cores with a linear characteristic and those with rectangular hysteresis loops are reviewed. Neeteson's approximation to the flux switching characteristic of the latter is applied to a circuit with the primary winding energized by a current step, and the secondary winding terminated in a series combination of inductance and resistance. After several changes of variable the resulting differential equation reduces to:

$$\frac{d^2u}{dt^2} + (C_1 + C_2 \sin u) \frac{du}{dt} = C_3 ,$$

where u is a function of the flux in the core, t is time, and the C's are constants. By assuming that $\sin u$ is constant over successive small time intervals, this equation is solved by means of the Laplace transform. An auxiliary differential equation for the secondary current is solved by utilizing an integrating factor and numerical integration. Curves of flux and current versus time are given for the cases:

$$C_1 = 1 ; C_2 = 0 , 1 , 2 , 4 ; C_3 = 20 .$$

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CONTENTS

ILLUSTRATIONS

Section 1
INTRODUCTION

By approximating with a semicircle the flux switching characteristic of a ferrite core possessing a rectangular hysteresis loop, Neeteson (Ref. 5), has simplified considerably the analysis of switching circuits containing such cores. Generally, the resulting differential equations are nonlinear. Neeteson has presented solutions for circuits in which a primary winding on the core is energized by a step current generator, and a secondary winding is terminated by resistance, by inductance, and by a parallel combination of resistance and inductance. One is led quite naturally to inquiring into the effect of terminating the secondary winding with a series combination of resistance and inductance. It will be shown that, after certain transformations, the resulting differential equation has the form

$$\frac{d^2u}{dt^2} + (C_1 + C_2 \sin u)\frac{du}{dt} = C_3$$

The flux in the core is a function of the variable u .

This paper is concerned principally with a method for solving numerically the foregoing differential equation. A few examples have been considered. The solutions are presented in the form of graphs for the variable u and for the output voltage and current, which are derivable from u by simple operations, time being the independent variable. Before delving into the method of solution, however, it is desirable to preface a few remarks concerning the differences in analytical treatments between cores with linear characteristics and cores with square hysteresis loops.

According to the well-known analogy between magnetic and electric circuits, the magneto-motive force in ampere-turns NI (MKS units), the reluctance R in ampere-turns per weber and the flux φ in webers are related by the equation (Ref. 6).

$$\mathrm{MMF} = \mathrm{NI} = \mathrm{R}\varphi . \tag{1}$$

Neeteson uses the symbol (AT) for NI, so that, following his notation, Eq. (1) becomes

$$(\mathrm{AT}) = \mathrm{R}\varphi . \tag{1a}$$

For the square-looped core being switched, the pertinent equation relating output voltage (which is equal to $d\varphi/dt$ in webers/sec) and the ampere turns (AT) is

$$\frac{d\varphi}{dt} = (\mathrm{AT}) \cdot r \cdot f(\varphi) = (\mathrm{AT}) \cdot r \cdot \sqrt{1 - (1 - \varphi/\varphi_m)^2} \tag{2}$$

where

r is a function of both the material and the shape of the core, and has the dimensions of resistance in ohms,

and: $f(\varphi) = \sqrt{1 - (1 - \varphi/\varphi_m)^2}$ is a nondimensional relation approximating the flux switching characteristic of the core in terms of the ratio of the flux ϕ which has been switched to φ_m, the latter being one-half the total flux capable of being switched.

Two figures, taken from Neeteson, will aid in clarifying the ideas involved. Figure 1-1 shows an idealized hysteresis loop for a core, and serves to define φ_m. Figure 1-2 shows the curve of a function derived by Lindsey (Ref. 4) which Neeteson approximated by the foregoing $f(\varphi)$. While the hysteresis loop of Fig. 1-1 pertains to the low-frequency alternating-current properties of the core, both curves of Fig. 1-2 concern the switching properties of the core to the extent described by Eq. (2).

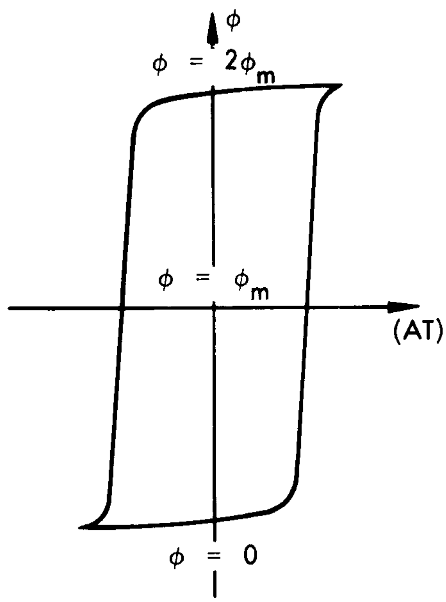


Fig. 1-1 Idealised Hysteresis Curve Illustrating the Meaning of φ and φ_m. (Taken from Neeteson.)

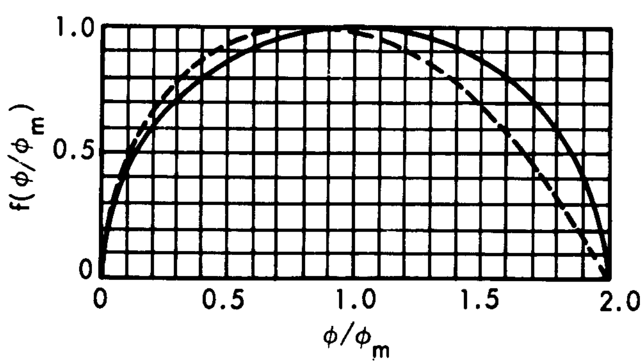


Fig. 1-2 Approximation of Lindsey's Curve (drawn in broken line) by a Semi-Circle. (Taken from Neeteson.)

The important point is that in Eq. (1) or (1a) the flux is proportional to the ampere-turns, whereas in Eq. (2) the rate of change of flux is proportional to the ampere-turns. Since $d\varphi/dt$ (in webers/second) is equal to a voltage, Eq. (2) actually states that the core acts like a nonlinear resistor, $r \cdot \sqrt{1 - (1 - \varphi/\varphi_m)^2}$, in contrast to the customary consideration of a winding on a magnetic material as an inductor.

The magnetic switching resistance r, which is a constant multiplier of the flux function, depends on the core material and geometry. Its defining equation is:

$$r = \rho \frac{A}{\ell} \text{ ohms}, \qquad \text{(core)} \qquad (3)$$

where:

ρ is the magnetic switching resistivity of the material in ohms per meter,

ℓ is the mean path length in meters,

A is the cross-sectional area in square meters.

This equation should be contrasted with the well-known formula for the electrical resistance of a wire of length ℓ and cross sectional area A:

$$r = \rho \frac{\ell}{A} = \text{ ohms}, \qquad \text{(wire)} \qquad (4)$$

where, now, ρ is the electrical resistivity of the material in ohm-meters. It is immediately evident that the geometrical terms A and ℓ for a core are inverted with respect to the terms for a wire; consequently the magnetic switching resistivity for a core has the dimension ohms per meter. It is interesting to see why this inversion exists.

The core resistance is defined as the ratio of the maximum pulse output voltage $(d\varphi/dt)_m$ to the effective input ampere turns (AT):

$$r = \frac{\left(\frac{d\varphi}{dt}\right)_m}{(AT)} \quad . \qquad (5)$$

Core resistance is measured experimentally as the slope of the straight line obtained by plotting the maximum pulse output voltage $(d\varphi/dt)_m$ versus the input ampere turns $N_1 I_1$, of the same form used to drive the circuit. The input turns $N_1 I_1$ are related to the effective input ampere turns (AT) by the relation

$$(AT) = N_1 I_1 - (AT)_0 , \tag{6}$$

where $(AT)_0$ is the threshold value below which no output voltage due to irreversible switching is obtained. See Fig. 1-3 (taken from Neeteson). Equations (5) and (6) may be combined to give:

$$\left(\frac{d\varphi}{dt}\right)_m = r(N_1 I_1) - r(AT)_0 \quad . \tag{7}$$

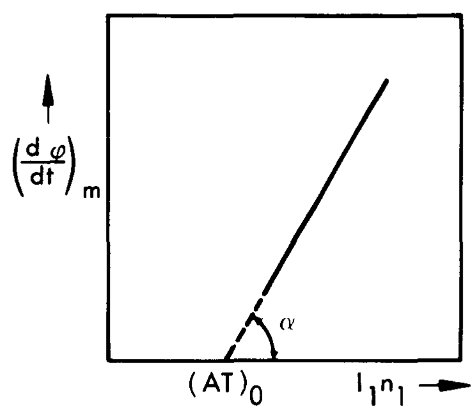


Fig. 1-3 Graph by Means of Which the Value of $r = \tan\alpha$ can be ascertained. (Taken from Neeteson.)

Upon taking the derivative of $(d\varphi/dt)_m$ with respect to $N_1 I_1$, (r and $(AT)_0$ are assumed to be constant with respect to time) we obtain:

$$r = \frac{d\left(\frac{d\varphi}{dt}\right)_m}{d(N_1 I_1)} \quad . \tag{8}$$

This expression is the slope of the foregoing mentioned straight line. The quantities on the right-hand side of Eq. (8) may be expressed in terms of the more fundamental quantities B and H (flux density and field intensity, respectively):

$$\left(\frac{d\varphi}{dt}\right)_m = \left(\frac{d\,BA}{dt}\right)_m = A\left(\frac{dB}{dt}\right)_m$$

and:

$$N_1 I_1 = H\ell \quad .$$

With these substitutions Eq. (8) becomes:

$$r = \frac{d\left[A\left(\frac{dB}{dt}\right)_m\right]}{d(H\ell)} = \frac{d\left(\frac{dB}{dt}\right)_m}{dH} \cdot \frac{A}{\ell} \quad . \tag{9}$$

The derivative $d\left[(dB/dt)_m\right]/dH$ is a function of the core material only. This quantity has been denoted by ρ:

$$\rho = \frac{d\left(\frac{dB}{dt}\right)_m}{dH} \quad . \tag{10}$$

Equation (9) becomes:

$$r = \rho\,\frac{A}{\ell}\ \text{ohms} \quad . \qquad \text{(core)}$$

The foregoing explanation, beginning wit'ı Eq. (5), accounts for the inversion of geometrical terms referred to previously.

Equation (2) is utilized in Section 2 to derive the differential equation for the circuit of concern. This is followed in Section 3 by the mathematics of the method for solving that differential equation. In Section 4 another differential equation relating the secondary current to the rate of change of flux is solved by means of an integrating factor and numerical integration. Computed curves are presented and discussed in Section 5. The paper is concluded with a short summary in Section 6.

Section 2

DERIVATION OF THE DIFFERENTIAL EQUATION

$$\frac{d^2u}{dt^2} + (C_1 + (C_2 \sin u) \frac{du}{dt} = C_3 .$$

The circuit to be analyzed is shown schematically in Fig. 2-1. Upon taking into account the secondary ampere-turns $N_2 I_2$, which tend to oppose the driving ampere-turns $N_1 I_1$, Eq. (2) of Section 1 is modified to become:

$$\frac{d\varphi}{dt} = r\sqrt{1 - (1 - \varphi/\varphi_m)^2} \left[N_1 I_1 - (AT)_0 - N_2 I_2\right] . \qquad (1)$$

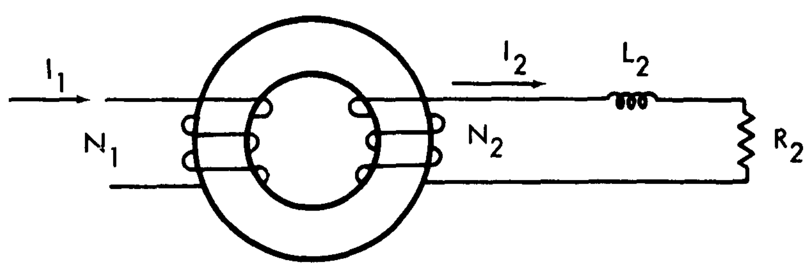


Fig. 2-1 Circuit of a Core Driven by a Current Generator I_1, with the Secondary Terminated by Inductance and Resistance in Series.

In order to find the secondary current we make use of Faraday's law that the voltage induced in the secondary winding by the change of flux in the core is $N_2\, d\varphi/dt$. This is equal to the sum of the voltage drops across inductor L_2 and resistor R_2:

$$N_2 \frac{d\varphi}{dt} = L_2 \frac{dI_2}{dt} + R_2 I_2 . \tag{2}$$

These two equations suffice to define the circuit behavior completely. The driving force $N_1 I_1$ is assumed to be a known function of time as, for example, a step function or a ramp. The maximum flux $2\varphi_m$ and the threshold value of ampere turns $(AT)_0$ are assumed to be known, as well as the values of L_2 , R_2 , N_1 , and N_2 . The quantities φ and I_2 are the unknowns. The problem consists in eliminating one of these, I_2 , for example, from either of the two equations, thereby obtaining an equation in terms of φ only. If the equation in φ alone can be solved, the answer can be substituted into Eq. (2) which is then solved for I_2 .

Proceeding along these lines, we first arrange Eq. (1) in the form:

$$I_2 = \frac{1}{N_2}\left[N_1 I_1 - (AT)_0 - \frac{1}{r\sqrt{1 - (1 - \varphi/\varphi_m)^2}}\frac{d\varphi}{dt}\right] . \tag{3}$$

By multiplying both sides of Eq. (3) by R_2, and by multiplying L_2 and taking the derivative with respect to time, we then obtain:

$$R_2 I_2 = \frac{R_2}{N_2}\left[N_1 I_1 - (AT)_0 - \frac{1}{r\sqrt{1 - (1 - \varphi/\varphi_m)^2}}\frac{d\varphi}{dt}\right] \tag{4}$$

$$L_2 \frac{dI_2}{dt} = \frac{L_2}{N_2}\left[N_1 \frac{dI_1}{dt} - \frac{d}{dt}\left(\frac{1}{r\sqrt{1 - (1 - \varphi/\varphi_m)^2}}\frac{d\varphi}{dt}\right)\right] . \tag{5}$$

Substitution of Eqs. (4) and (5) into Eq. (2) gives:

$$N_2 \frac{d\varphi}{dt} = \frac{L_2}{N_2}\left[N_1 \frac{dI_1}{dt} - \frac{d}{dt}\left(\frac{1}{r\sqrt{1-(1-\varphi/\varphi_m)^2}}\frac{d\varphi}{dt}\right)\right] + \frac{R_2}{N_2}\left[N_1 I_1 - (AT)_0 - \frac{1}{r\sqrt{1-(1-\varphi/\varphi_m)^2}}\frac{d\varphi}{dt}\right] . \qquad (6)$$

After similar terms are grouped, the following equation results:

$$\frac{L_2}{N_2 r}\frac{d}{dt}\left(\frac{1}{\sqrt{1-(1-\varphi/\varphi_m)^2}}\frac{d\varphi}{dt}\right) + \left(N_2 + \frac{R_2}{N_2 r}\frac{1}{\sqrt{1-(1-\varphi/\varphi_m)^2}}\right)\frac{d\varphi}{dt} = \frac{R_2}{N_2}\left[N_1 I_1 - (AT)_0\right] + L_2 \frac{N_1}{N_2}\frac{dI_1}{dt} \qquad (7)$$

Now, let:

$$\frac{\varphi}{\varphi_m} = x$$

by means of which we obtain

$$\frac{d\varphi}{dt} = \varphi_m \frac{dx}{dt}$$

and:

$$\sqrt{1-(1-\varphi/\varphi_m)^2} = \sqrt{1-(1-x)^2} .$$

With these substitutions Eq. (7) becomes:

$$\frac{L_2\varphi_m}{N_2 r}\frac{d}{dt}\left(\frac{1}{\sqrt{1-(1-x)^2}}\frac{dx}{dt}\right)+\left(N_2\varphi_m+\frac{R_2\varphi_m}{N_2 r}\frac{1}{\sqrt{1-(1-x)^2}}\right)\frac{dx}{dt}$$

$$=\frac{R_2}{N_2}\left[N_1 I_1-(AT)_0\right]+L_2\frac{N_1}{N_2}\frac{dI_1}{dt} \quad . \tag{8}$$

Next, let:

$$1 - x = y$$

whence:

$$-\frac{dx}{dt}=\frac{dy}{dt} \quad .$$

Substitution of these into Eq. (8) yields:

$$-\frac{L_2\varphi_m}{N_2 r}\frac{d}{dt}\left(\frac{1}{\sqrt{1-y^2}}\frac{dy}{dt}\right)-\left(N_2\varphi_m+\frac{R_2\varphi_m}{N_2 r}\frac{1}{\sqrt{1-y^2}}\right)\frac{dy}{dt}$$

$$=\frac{R_2}{N_2}\left[N_1 I_1-(AT)_0\right]+L_2\frac{N_1}{N_2}\frac{dI_1}{dt} \quad . \tag{9}$$

Lastly, let:

$$y = \cos u$$

whence:

$$\frac{dy}{dt} = - \sin u \frac{du}{dt} \;,$$

$$\sqrt{1 - y^2} = \sin u \,,$$

and:

$$- \frac{d}{dt}\left(\frac{1}{\sqrt{1 - y^2}} \frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{\sin u}{\sin u} \frac{du}{dt}\right) = \frac{d^2 u}{dt^2}$$

With this new change of variable Eq. (9) is transformed into:

$$\frac{L_2 \varphi_m}{N_2 r} \frac{d^2 u}{dt^2} + \left(\frac{R_2 \varphi_m}{N_2 r} + N_2 \varphi_m \sin u\right) \frac{du}{dt}$$

$$= \frac{R_2}{L_2} \left[N_1 I_1 - (AT)_0\right] + L_2 \frac{N_1}{N_2} \frac{dI_1}{dt} \tag{10}$$

Upon dividing through by $L_2 \varphi_m / N_2 r$ this becomes

$$\frac{d^2 u}{dt^2} + \left(\frac{R_2}{L_2} + N_2^2 \frac{r}{L_2} \sin u\right) \frac{du}{dt}$$

$$= \frac{R_2 r}{L_2 \varphi_m} \left[N_1 I_1 - (AT)_0\right] + \frac{N_1 r}{\varphi_m} \frac{dI_1}{dt} \tag{11}$$

For the particular case in which the driving force I_1 is a step function, its derivative with respect to time dI_1/dt is equal to 0 for $t \geq 0+$, whereupon Eq. (11) reduces to:

$$\frac{d^2u}{dt^2} + \left(\frac{R_2}{L_2} + N_2^2 \frac{r}{L_2} \sin u\right) \frac{du}{dt} = \frac{R_2 r}{L_2 \varphi_m} \left[N_1 I_1 - (AT)_0\right] . \qquad (12)$$

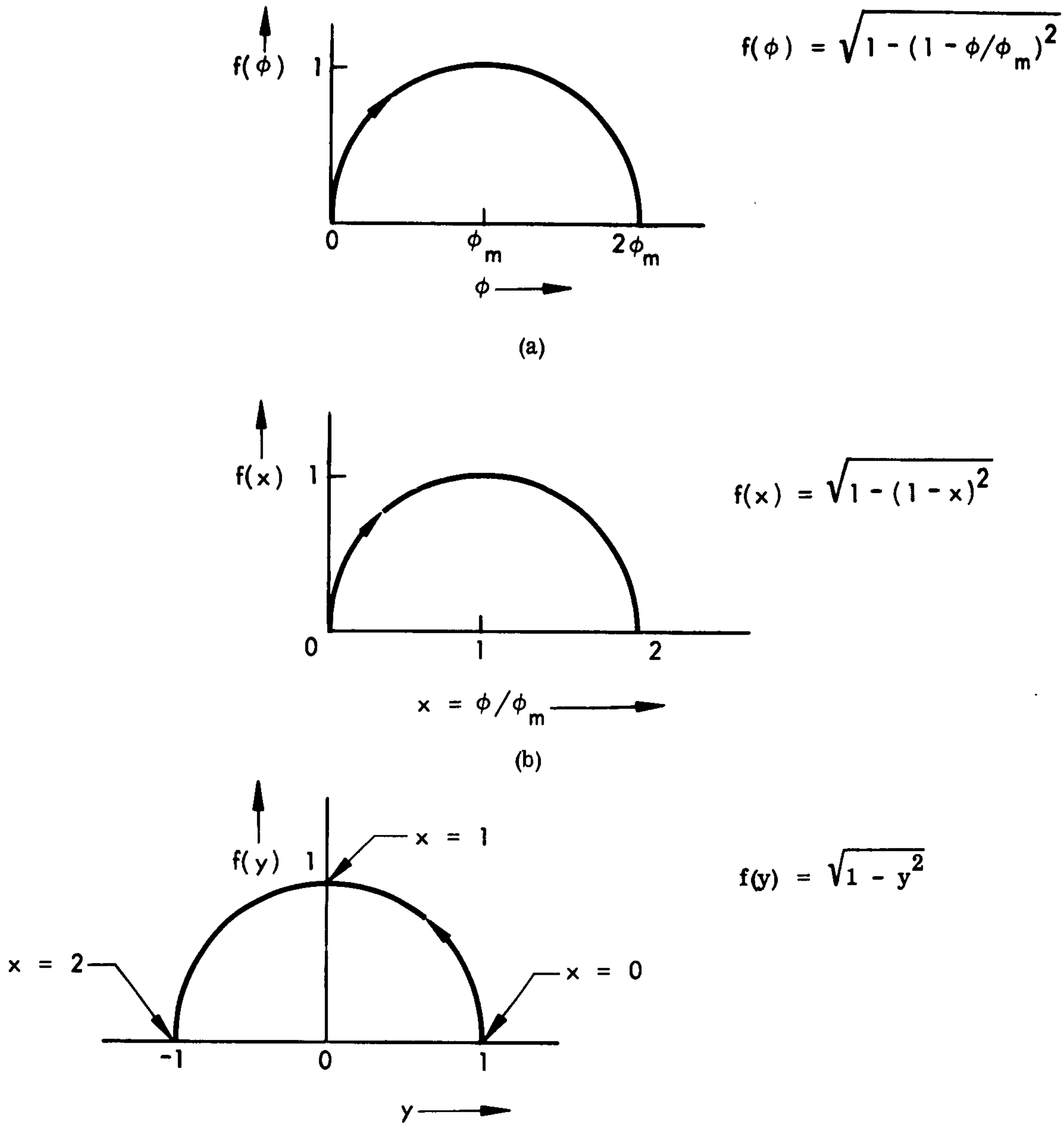
This last equation has the form indicated in the introduction. Before presenting its solution, we believe it worthwhile to review the transformations leading to the final form.

Figures 2-2a, 2-2b, and 2-2c show plots of the function $f(\varphi) = \sqrt{1 - (1 - \varphi/\varphi_m)^2}$ before and after being subjected to the changes of variable: $x = \varphi/\varphi_m$, $y = 1 - x$. The arrows on the curves indicate the direction taken by a moving point as φ increases. In Figs. 2-2a and 2-2b this motion is clockwise, while in Fig. 2-2c this motion is counterclockwise. The contrast of directions may also be seen by tabulating corresponding values for φ, x and y as follows:

φ	x	y
0	0	1
φ_m	1	0
$2\varphi_m$	2	- 1

Now consider the two functions $y = \cos u$, and $f(u) = \sqrt{1 - \cos^2 u} = \sin u$, over the same range of variables, namely:

$y = \cos u$	u	$f(u) = \sin u$
1	0	0
0	$\pi/2$	1
- 1	π	0



$f(\phi)$ versus ϕ ; $f(x)$ versus x ; $f(y)$ versus y

(c)

Fig. 2-2 Plots of the Function $f(\phi) = \sqrt{1 - (1 - \phi/\phi_m)^2}$

Plots of these functions are given in Figs. 2-3a and 2-3b. Again the arrows on the curves indicate the direction taken by a moving point as φ increases. Attention is called to the fact that, as φ increases from 0 through φ_m to $2\varphi_m$, the variable u increases from 0 through $\pi/2$ to π.

It should be kept in mind that φ is the dependent variable, whereas time is the independent variable. The values $\varphi = 0$ and $u = 0$ correspond to $t = 0$, and increasing values of φ and u are associated with increasing time. When $t = 0$, then $\sin u = 0$, and the differential Eq. (12) becomes:

$$\frac{d^2u}{dt^2} + \frac{R_2}{L_2}\frac{du}{dt} = \frac{R_2 r}{L_2 \varphi_m}\left[N_1 I_1 - (AT)_0\right] . \qquad (13)$$

At the time when $\varphi = \varphi_m$, $\sin u = 1$, and the differential equation, therefore, has the form:

$$\frac{d^2u}{dt^2} + \left(\frac{R_2}{L_2} + N_2^2 \frac{r}{L_2}\right)\frac{du}{dt} = \frac{R_2 r}{L_2 \varphi_m}\left[N_1 I_1 - (AT)_0\right] . \qquad (14)$$

At the time when $\varphi = 2\varphi_m$, the time when the switching process ends, we again have $\sin u = 0$; consequently, the differential equation once more reduces to:

$$\frac{d^2u}{dt^2} + \frac{R_2}{L_2}\frac{du}{dt} = \frac{R_2 r}{L_2 \varphi_m}\left[N_1 I_1 - (AT)_0\right] . \qquad (15)$$

It is not difficult to find an electrical analogue for Eq. (12). Assume a circuit consisting of a battery $\frac{R_2 r}{\varphi_m}\left[N_1 I_1 - (AT)_0\right]$ energizing a circuit comprised of an inductance L_2 in series with a fixed resistor R_2 and a variable resistance. Let

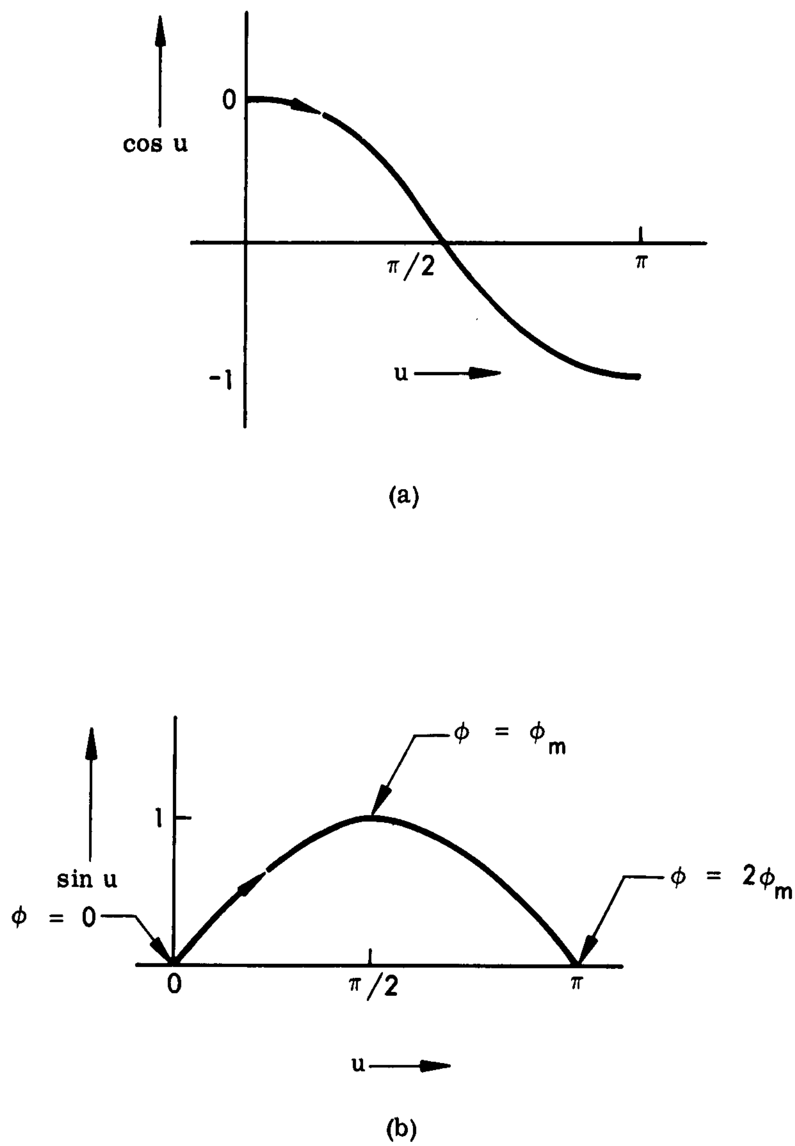


Fig. 2.3 Plots of Functions $y = \cos u$, and $f(u) = \sqrt{1 - \cos^2 u}$

the variable resistance be a sinusoidal function of the charge q which has flowed through the circuit and be of maximum value $N_2^2 r$, namely, $N_2^2 r \sin q$. The circuit is shown schematically in Fig. 2-4.

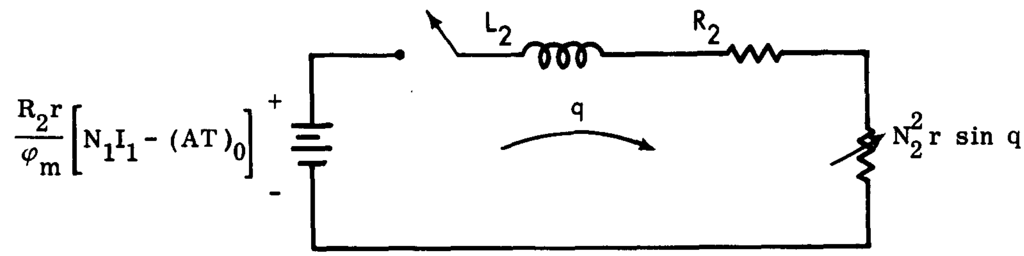


Fig. 2-4 Electric Circuit Analogue for Eq. (12)

The appropriate electrical differential equation is:

$$\frac{d^2 q}{dt^2} + \left(\frac{R_2}{L_2} + N_2^2 \frac{r}{L_2} \sin q\right) \frac{dq}{dt} = \frac{R_2 r}{L_2 \varphi_m} \left[N_1 I_1 - (AT)_0\right] . \tag{16}$$

As time progresses from $t = 0$ (at which time $q = 0$), through the time when $q = \pi/2$ coulombs to the time at which $q = \pi$ coulombs, the variable resistance ranges from 0 to $N_2^2 r$ and back to 0.

At any given instant of time, Eq. (12) may be considered as a second order differential equation with constant coefficients. At a slightly later instant of time Eq. (12) may again be considered to have constant coefficients, but with the coefficient of the first derivative a little altered from its previous value. This suggests that a possible method of solution is to solve the differential equation for each particular instant of time by a standard method as, for example, by the Laplace transform, using as the initial conditions for each solution the value of the dependent variable and its first derivative obtained from the solution for a slightly prior time. This solution was completed and will be outlined in Section 3. For that solution we need first to settle upon a value for du/dt at $t = 0$, which will occupy us for the rest of Section 2. (The value $u = 0$ at $t = 0$ is the second initial condition needed, but this is already known.)

Lacking any definite knowledge about du/dt at $t = 0$, let us assume that it is equal to zero. We shall now show that this assumption implies that in the vicinity of $t = 0$ the flux φ must vary as a power of t greater than 2. Since in view of experimental work this is reasonable, the assumption will be used in the later treatment.

From the relation:

$$u = \text{arc cos } y$$

we get, by differentiation:

$$\frac{du}{dt} = -\frac{1}{\sqrt{1 - y^2}} \frac{dy}{dt} = -\frac{1}{\sqrt{1 - y}\,\sqrt{1 + y}} \frac{dy}{dt} . \tag{17}$$

At $t = 0$, $y = 1$, and the denominator of Eq. (17) is zero. In order for du/dt to remain finite at this point, we must have $\frac{dy}{dt} \to 0$ as a greater power of $(1-y)$ than $\frac{1}{2}$, say as $(1-y)^{\frac{1}{2}+\epsilon}$ where, for reasons which will appear presently, the range of is restricted to $0 < \epsilon < \frac{1}{2}$. From the original transformations:

$$1 - y = x = \frac{\varphi}{\varphi_m} ,$$

the assumption that:

$$\frac{dy}{dt} \to 0 \quad \text{as} \quad (1 - y)^{\frac{1}{2} + \epsilon}$$

requires that:

$$\frac{d\varphi}{dt} \to 0 \quad \text{as} \quad \frac{\varphi^{\frac{1}{2}+\epsilon}}{\varphi_m} \quad .$$

Therefore let

$$\frac{d\varphi}{dt} = K_1 \varphi^{\frac{1}{2}+\epsilon} \qquad (K_1 = \text{a constant}) \tag{18}$$

or alternatively:

$$\frac{\varphi}{\varphi^{\frac{1}{2}+\epsilon}} = K_1 \, dt \quad . \tag{19}$$

An indefinite integral yields:

$$\varphi^{\frac{1}{2}-\epsilon} = K_1 t + K_2 \qquad (K_2 = \text{a constant}) \tag{20}$$

whence:

$$\varphi = (K_1 t + K_2)^{1/\left(\frac{1}{2}-\epsilon\right)} = (K_1 t + K_2)^{2/(1-2\epsilon)} \quad . \tag{21}$$

When $t = 0$, $\varphi = 0$; therefore, $K_2 = 0$. Consequently, we get:

$$\varphi = K_1 t^{\frac{2}{1-2\epsilon}} \quad . \tag{22}$$

Since ϵ is positive, φ varies as a power of t greater than 2 in the vicinity of $t = 0$.

If $\epsilon = 1/2$, we must go back to Eq. (19) and repeat the integration. We then obtain:

$$\frac{d\varphi}{\varphi} = K_1 \, dt \tag{23}$$

$$\ell n \, \varphi = K_1 t + \ell n \, K_2 \tag{24}$$

whence:

$$\varphi = K_2 \exp (K_1 t) \quad . \tag{25}$$

The value $t = 0$ yields $\varphi = K_2$, which is inconsistent with the previously assigned condition $\varphi = 0$. Therefore, the value $\epsilon = 1/2$ is not allowed.

For values of $\epsilon > 1/2$, the exponent of $(K_1 t + K_2)$ in Eq. (21) is negative and the following results:

$$\varphi = \frac{1}{(K_1 t + K_2)^{\frac{2}{2\epsilon - 1}}} \quad .$$

The value $t = 0$ yields

$$\varphi = \frac{1}{K_2^{\frac{2}{2\epsilon - 1}}} \quad ,$$

which again is inconsistent with the assigned condition $\varphi = 0$. Therefore, the values $\epsilon > 1/2$ also are not allowed. Consequently, the range of ϵ compatible with the conditions $\varphi = 0$, $d\varphi/dt = 0$ at $t = 0$ (or $u = 0$, $du/dt = 0$ at $t = 0$) is:

$$0 > \epsilon > \frac{1}{2} \; ,$$

as mentioned above. The value of ϵ will not be determined precisely, although it could be determined from a particular solution. The discussion on the range of ϵ merely serves to give us an idea of the shape of the flux curve near the time origin, that is, it is at least parabolic.

Section 3
ITERATIVE PROCEDURE FOR SOLVING THE DIFFERENTIAL EQUATION

For reference we repeat Eq. (12) of Section 2:

$$\frac{d^2u}{dt^2} + \left(\frac{R_2}{L_2} + N_2^2 \frac{r}{L_2} \sin u\right)\frac{du}{dt} = \frac{R_2}{L_2}\frac{r}{\varphi_m}\left[N_1 I_1 - (AT)_0\right] . \tag{1}$$

For brevity we shall make the further substitutions:

$$A_i = \frac{R_2}{L_2} + N_2^2 \frac{r}{L_2} \sin u(t_i)$$

$$B = \frac{R_2}{L_2}\frac{r}{\varphi_m}\left[N_1 I_1 - (AT)_0\right] .$$

In these substitutions, B is a constant. The subscript i of A_i and $\sin u(t_i)$ is used to indicate that these quantities, which are variable, will be evaluated at the discrete times $t_i (i = 0, 1, 2, \ldots)$, and considered constant over the interval $t_{i+1} - t_i$. In addition to $u(t_i)$ we must evaluate its first derivative with respect to time $u'(t_i)$. With this new notation Eq. (1) becomes:

$$\frac{d^2u}{dt^2} + A_i \frac{du}{dt} = B \tag{2}$$

subject to the initial conditions

$$u(t_0) = 0$$
$$u'(t_0) = 0 .$$

Since we are considering A_i as a constant for the present, we may apply the Laplace transform to Eq. (2). To do this we need the following relations (Ref. 1):

$$\mathcal{L}\,\frac{d^2u}{dt^2} = s^2U(s) - su(t_i) - u'(t_i) \tag{3a}$$

$$\mathcal{L}\,\frac{du}{dt} = sU(s) - u(t_i) \tag{3b}$$

$$\mathcal{L}\,B = \frac{B}{s}\,, \tag{3c}$$

where $U(s)$ is the Laplace transform of $u(t)$. The use of $u(t_i)$ and $u'(t_i)$ in Eq. (3) rather than $u(t_0)$ or $u'(t_0)$ [or $u(0+)$ and $u'(0+)$ in more customary notation] serves to emphasize the fact that time is measured from t_i $(i = 0, 1, 2, \ldots)$ rather than only from t_0. Substitution of Eq. (3) into Eq. (2) yields:

$$s^2U(s) - su(t_i) - u'(t_i) + A_i\,[sU(s) - u(t_i)] = \frac{B}{s}\,. \tag{4}$$

Rearranging the terms of Eq. (4) we obtain:

$$\left(s^2 + A_i s\right)U(s) = \frac{B}{s} + u'(t_i) + A_i u(t_i) + su(t_i) \tag{5}$$

whence:

$$U(s) = \frac{B}{s^2(s + A_i)} + \frac{u'(t_i) + A_i u(t_i)}{s(s + A_i)} + \frac{u(t_i)}{s + A_i}\,.$$

From a table of Laplace transforms (Ref. 2) we find the following:

$$\mathcal{L}^{-1} \frac{1}{s^2(s + A_i)} = \frac{\exp(-A_i t) + A_i t - 1}{A_i^2} \tag{7a}$$

$$\mathcal{L}^{-1} \frac{1}{s(s + A_i)} = \frac{1 - \exp(-A_i t)}{A_i} \tag{7b}$$

$$\mathcal{L}^{-1} \frac{1}{s + A_i} = \exp(-A_i t) \tag{7c}$$

With the aid of these, the inverse transform of Eq. (6) is:

$$u(t) = \frac{B}{A_i^2}\left[\exp(-A_i t) + A_i t - 1\right] + \left[\frac{u'(t_i) + A_i u(t_i)}{A_i}\right]\left[1 - \exp(-A_i t)\right] + u(t_i)\exp(-A_i t) \tag{8}$$

Multiplying out the second group of terms on the right and combining it with the third we obtain:

$$u(t) = u(t_i) + \frac{u'(t_i)}{A_i}\left[1 - \exp(-A_i t)\right] + \frac{B}{A_i^2}\left[\exp(-A_i t) + A_i t - 1\right] \tag{9}$$

Differentiation of Eq. (9) yields:

$$u'(t) = u'(t_i)\exp(-A_i t) + \frac{B}{A_i}\left[1 - \exp(-A_i t)\right] \tag{10}$$

The forms of these equations are interesting. Let us first consider Eq. (9). This equation states that the value of u at some time t measured from time t_i, namely $u(t)$, depends on the value of u at the prior time $u(t_i)$; on its derivative at that time $u'(t_i)$ multiplied by $[1 - \exp(-A_i t)]/A_i$; and on the driving force B multiplied by $[\exp(-A_i t) + A_i t - 1]/A_i^2$. Equation (10) says that the derivative of u at some time t measured from time t_i, namely $u'(t)$, depends on the value of u' at the prior time $u'(t_i)$, multiplied by $\exp(-A_i t)$, and on the driving force B multiplied by $[1 - \exp(-A_i t)]/A_i$. These equations are assumed to be valid only over a small time interval $\Delta t = t_{i+1} - t_i$. These equations, therefore, should more properly be written as

$$u(t_{i+1}) = u(t_i) + \frac{u'(t_i)}{A_i}\left\{1 - \exp\left[-A_i(t_{i+1} - t_i)\right]\right\} + \frac{B}{A_i^2}\left\{\exp\left[-A_i(t_{i+1} - t_i)\right] + A_i(t_{i+1} - t_i) - 1\right\} \tag{11}$$

and:

$$u'(t_{i+1}) = u'(t_i)\exp\left[-A_i(t_{i+1} - t_i)\right] + \frac{B}{A_i}\left\{1 - \exp\left[-A_i(t_{i+1} - t_i)\right]\right\}. \tag{12}$$

From the foregoing equations, it is apparent that if we know the values of u, u' and A at time t_i, namely $u(t_i)$, $u'(t_i)$, and A_i, we can then use Eqs. (11), (12) and the following relation:

$$A_i = \frac{R_2}{L_2} + N_2^2 \frac{r}{L_2} \sin u(t_i) \tag{13}$$

to calculate $u(t_{i+1})$, $u'(t_{i+1})$ and A_{i+1}. These last equations may then be utilized to calculate $u(t_{i+2})$, $u'(t_{i+2})$ and A_{i+2}, etc. Proceeding in this way, we are able to calculate the solution of Eq. (1), starting with $u(t_0) = 0$, and continuing until $u(t_i) = \pi$.

This procedure lends itself readily to machine or hand computation. It will be noticed that the expression multipling $u'(t_i)$ in Eq. (11) is identical with that multiplying B in Eq. (12). Consequently, only three exponential expressions need be evaluated at each step.

If in Eqs. (11) and (12) we denote the time interval by $\Delta t = t_{i+1} - t_i$, and if we choose this interval small enough so that $A_i \Delta t \ll 1$, then it is convenient to expand the exponential expressions in infinite series of which only the first few terms will be needed. We then obtain:

$$\frac{1}{A_i}\left[1 - \exp(-A_i \Delta t)\right] = \frac{1}{A_i}\left\{1 - \left[1 - A_i \Delta t + \frac{(A_i \Delta t)^2}{2!} - \frac{(A_i \Delta t)^3}{3!} + \cdots\right]\right\}$$

$$= \frac{1}{A_i}\left[A_i \Delta t - \frac{(A_i \Delta t)^2}{2!} + \frac{(A_i \Delta t)^3}{3!} - \cdots\right]$$

$$= \Delta t\left[1 - \frac{A_i \Delta t}{2!} + \frac{(A_i \Delta t)^2}{3!} - \cdots\right] \tag{14a}$$

$$\frac{1}{A_i^2}\left[\exp(-A_i \Delta t) + A_i \Delta t - 1\right]$$

$$= \frac{1}{A_i^2}\left[1 - A_i \Delta t + \frac{(A_i \Delta t)^2}{2!} - \frac{(A_i \Delta t)^3}{3!} + \frac{(A_i \Delta t)^4}{4!} - \cdots + A_i \Delta t - 1\right]$$

$$= \frac{1}{A_i^2}\left[\frac{(A_i \Delta t)^2}{2!} - \frac{(A_i \Delta t)^3}{3!} + \frac{(A_i \Delta t)^4}{4!} - \cdots\right]$$

$$= (\Delta t)^2\left[\frac{1}{2!} - \frac{A_i \Delta t}{3!} + \frac{(A_i \Delta t)^2}{4!} - \cdots\right] \tag{14b}$$

$$\exp(-A_i\Delta t) = 1 - A_i\Delta t + \frac{(A_i\Delta t)^2}{2!} - \frac{(A_i\Delta t)^3}{3!} + \cdots \tag{14c}$$

With these substitutions, Eqs. (11) and (12) become:

$$u(t_{i+1}) = u(t_i) + u'(t_i)\Delta t\left[1 - \frac{A_i\Delta t}{2!} + \frac{(A_i\Delta t)^2}{3!} - \frac{(A_i\Delta t)^3}{4!} + \cdots\right]$$

$$+ B(\Delta t)^2\left[\frac{1}{2!} - \frac{A_i\Delta t}{3!} + \frac{(A_i\Delta t)^2}{4!} - \frac{(A_i\Delta t)^3}{5!} + \cdots\right] \tag{15}$$

and:

$$u'(t_{i+1}) = u'(t_i)\left[1 - A_i\Delta t + \frac{(A_i\Delta t)^2}{2!} - \frac{(A_i\Delta t)^3}{3!} + \cdots\right]$$

$$+ B\Delta t\left[1 - \frac{A_i\Delta t}{2!} + \frac{(A_i\Delta t)^2}{3!} - \frac{(A_i\Delta t)^3}{4!} + \cdots\right] \tag{16}$$

As an example of the use of these formulas, consider the following equation:

$$\frac{d^2u}{dt^2} + (1 + \sin u)\frac{du}{dt} = 20 \tag{17}$$

Here we have set:

$$\frac{R_2}{L_2} = 1 \text{ second}^{-1}$$

$$N_2^2 \frac{r}{L_2} = 1 \text{ second}^{-1}$$

$$B = \frac{R_2}{L_2} \frac{r}{\varphi_m} \left[N_1 I_1 - (AT)_0 \right] = 20 \text{ seconds}^{-2} .$$

Then:

$$A_i = 1 + \sin u(t_i) \qquad (i = 0, 1, 2, \ldots) \; .$$

As u varies from 0 through $\pi/2$ to π, the quantity A_i varies from 1 through 2 and back to 1 again. Let us choose a time interval $\Delta t = 0.05$ seconds, which is small compared to the time constants L_2/R_2 and $L_2/N_2^2 r$, each of which is equal to 1 second. We also note that $A_i \Delta t$ varies between 0.05 and 0.1 which are small compared to 1, so that only a few terms of the expansions in Eqs. (15) and (16) are required. Utilizing the conditions

$$u(t_0) = 0$$

$$u'(t_0) = 0$$

we have that:

$$A_0 = 1$$

$$A_0 \Delta t = 0.05$$

and that:

$$u(t_1) = u(0.05) = 0 + 0 + 20(0.05)^2 (0.491\,770) = 0.024\,589$$

$$u'(t_1) = u'(0.05) = 0 + 20(0.05)(0.975\,412) = 0.975\,412$$

$$\sin u(t_1) = \sin(0.024589) = 0.024\ 587$$

$$A_1 = 1 + 0.024\ 587 = 1.024\ 587$$

For the next set of computations we then get:

$$A_1 \Delta t = 0.051\ 229$$

$$u(t_2) = u(0.10) = 0.024\ 589 + 0.975\ 412(0.05)(0.974\ 817) + 20(0.05)^2(0.491\ 570) = 0.096\ 710$$

$$u'(t_2) = u'(0.10) = 0.975\ 412(0.950\ 061) + 20(0.05)(0.05)(0.974\ 817) = 1.901\ 518$$

and so forth.

From the solutions for u and u' we are then able to determine the flux and its rate of change by going back through the transformations:

$$1 - y = x = \frac{\varphi}{\varphi_m} \quad , \quad y = \cos u \ .$$

There results:

$$\frac{\varphi}{\varphi_m} = 1 - \cos u \tag{18}$$

and:

$$\frac{1}{\varphi_m} \frac{d\varphi}{dt} = \sin u \frac{du}{dt} = \sin u\,[u'(t)]$$

in which φ_m is assumed to be known.

Section 4

SOLUTION OF THE DIFFERENTIAL EQUATIONS FOR THE SECONDARY CURRENT

In order to determine the secondary current we have to solve Eq. (2) of Section 2 which is rewritten in the form:

$$\frac{dI_2}{dt} + \frac{R_2}{L_2} I_2 = \frac{N_2}{L_2} \frac{d\varphi}{dt} . \tag{1}$$

Here $\frac{d\varphi}{dt}$ is known from Section 3, but not in a neat analytical form. Equation (1) is a particular case of a more general equation whose solution is readily available (Ref. 7):

$$\frac{dI_2}{dt} + P(t) I_2 = Q(t) \tag{2}$$

Upon multiplying both sides of this equation by the integrating factor $\exp [\int P(t) dt]$, the left-hand side becomes an exact differential and results in the following:

$$\frac{d}{dt} \left\{ I_2 \exp [\int P(t) dt] \right\} = Q(t) \exp [\int P(t) dt] . \tag{3}$$

Integration of Eq. (3) then gives:

$$I_2 \exp [\int P(t) dt] = \int Q(t) \exp [\int P(t) dt] dt + C \tag{4}$$

where C is a constant.

Applying this method to Eq. (1) with:

$$P(t) = \frac{R_2}{L_2}$$

$$Q(t) = \frac{N_2}{L_2}\frac{d\varphi}{dt}$$

we obtain the counterpart to Eq. (3):

$$\frac{d}{dt}\left[I_2 \exp\left(\frac{R_2}{L_2}\,t\right)\right] = \frac{N_2}{L_2}\frac{d\varphi}{dt}\exp\left(\frac{R_2}{L_2}\,t\right) \tag{5}$$

In order to avoid the integration constant C which appeared in Eq. (4), we shall integrate Eq. (5) between the limits 0 and t , rather than use an indefinite integral. We then obtain:

$$I_2 \exp\left(\frac{R_2}{L_2}\,t\right)\Bigg]_0^t = \frac{N_2}{L_2}\int_0^t \frac{d\varphi}{dt}\exp\left(\frac{R_2}{L_2}\,t\right)dt \tag{6}$$

from which:

$$I_2(t)\exp\left(\frac{R_2}{L_2}\,t\right) - I_2(0) = \frac{N_2}{L_2}\int_0^t \frac{d\varphi}{dt}\exp\left(\frac{R_2}{L_2}\,t\right)dt \tag{7}$$

where the notation $I_2(t)$ is used to emphasize that I_2 is a function of time, and $I_2(0)$ is the current I_2 at $t = 0$. Solving for $I_2(t)$ we then find:

$$I_2(t) = I_2(0)\exp\left(-\frac{R_2}{L_2}\,t\right) + \frac{N_2}{L_2}\exp\left(-\frac{R_2}{L_2}\,t\right)\int_0^t \frac{d\varphi}{dt}\exp\left(\frac{R_2}{L_2}\,t\right)dt\,. \tag{8}$$

Since in our case $I_2(0) = 0$, Eq. (8) reduces to:

$$I_2(t) = \frac{N_2}{L_2} \exp\left(-\frac{R_2}{L_2} t\right) \int_0^t \frac{d\varphi}{dt} \exp\left(\frac{R_2}{L_2} t\right) dt . \tag{9}$$

It will be more convenient to obtain the solution in terms of $\frac{dx}{dt} = \frac{1}{\varphi_m} \frac{d\varphi}{dt}$, which is easily done as follows:

$$I_2(t) = N_2 \frac{\varphi_m}{L_2} \exp\left(-\frac{R_2}{L_2} t\right) \int_0^t \frac{dx}{dt} \exp\left(\frac{R_2}{L_2} t\right) dt . \tag{10}$$

This foregoing equation may be normalized by dividing through by $N_2 \varphi_m / L_2$ to give:

$$\frac{I_2(t)}{N_2 \varphi_m / L_2} = \exp\left(-\frac{R_2}{L_2} t\right) \int_0^t \frac{dx}{dt} \exp\left(\frac{R_2}{L_2} t\right) dt . \tag{11}$$

The integral on the right in Eq. (11) was evaluated by using the trapezoidal rule (Ref. 7) which is simple to apply and well suited to provide a "running" value of the integral. Multiplying this running value of the integral by $\exp[-(R_2/L_2)t]$ gives the normalized current. By way of illustration, for the differential equation (17) used as an example in Section 3, the right-hand side of Eq. (11) becomes:

$$e^{-t} \int_0^t \frac{dx}{dt} e^t dt .$$

The first few values of $\frac{dx}{dt}$ arrived at by the iterative procedure of Section 3, and of

$\frac{dx}{dt} e^t$, $\int_0^t \frac{dx}{dt} e^t dt$, and $e^{-t} \int_0^t \frac{dx}{dt} e^t dt$ are as follows:

t	$\frac{dx}{dt}$	$\frac{dx}{dt} e^t$	$\int_0^t \frac{dx}{dt} e^t dt$	$e^{-t} \int_0^t \frac{dx}{dt} e^t dt$
0	0	0	0	0
0.05	0.023 982	0.025 212	0.000 630	0.000 599
0.10	0.183 609	0.202 919	0.006 334	0.005 731
0.15	0.587 380	0.682 438	0.028 468	0.024 503

The integration is carried out from $t = 0$ to the time at which $u = \pi$. For this example, that time is approximately 0.65 seconds, at which time the switching process is complete, the flux φ has reached the value $2\varphi_m$, and thereafter $\frac{d\varphi}{dt} = 0$.

The current in the secondary circuit has some finite value, and must decay to zero through the inductance L_2 , the resistance R_2 , and the secondary turns N_2 linking the core. Since $\frac{d\varphi}{dt}$ is zero, and will be assumed as remaining zero throughout the current decay time, there is no voltage induced in the secondary. The secondary winding, consequently, may be assumed as a short circuit during this time. The current decays, therefore, from its previous final value I_{2f} in accordance with the differential equation:

$$L_2 \frac{dI_2}{dt} + R_2 I_2 = 0 \qquad (12)$$

The solution is known to be (Ref. 3):

$$I_2(t) = I_{2_f} \exp\left(-\frac{R_2}{L_2} t\right) . \tag{13}$$

where time t is now measured from the time at which the switching process reached completion. In order to keep the quantities involved consistent with the prior notation, this equation may be normalized also and written as:

$$\frac{I_2(t)}{N_2\varphi_m/L_2} = \frac{I_{2_f}}{N_2\varphi_m/L_2} \exp\left(-\frac{R_2}{L_2} t\right) . \tag{14}$$

For the example we are pursuing, the final normalized current was 1.595 at $t = 0.65$ seconds. Thereafter the current was assumed to decay as:

$$\frac{I_2(t)}{N_2\varphi_m/L_2} = 1.595\, e^{-t} .$$

Section 5
COMPUTED CURVES

The following differential equations were solved by the methods outlined previously.

$$\frac{d^2u}{dt^2} + \frac{du}{dt} = 20 \quad (1)$$

$$\frac{d^2u}{dt^2} + (1 + \sin u)\frac{du}{dt} = 20 \quad (2)$$

$$\frac{d^2u}{dt^2} + (1 + 2\sin u)\frac{du}{dt} = 20 \quad (3)$$

$$\frac{d^2u}{dt^2} + (1 + 4\sin u)\frac{du}{dt} = 20 \quad (4)$$

Curves were plotted for the quantities $u(t)$, $u'(t)$, $x(t)$, $x'(t)$ and $I_2(t)$, with $I_2(t)$ presented on two different time scales in order to show the large trailing edge of the current. These are displayed for Eq. (1) in Figs. 5-1a, 5-1b, 5-1c; for Eq. (2) in Figs. 5-2a, 5-2b, 5-2c; for Eq. (3) in Figs. 5-3a, 5-3b, 5-3c; and for Eq. (4) in Figs. 5-4a, 5-4b, 5-4c. The ordinate and abscissa corresponding to the time when the variable $u = \pi$, which ends the switching process, are indicated on Figs. 5-1a, 5-2a, 5-3a, and 5-4a. As the coefficient of $\sin u$ in the differential equations varies from 0 through 4, the switching time increased from 0.615 through 0.655 and 0.705 to 0.830 seconds.

The reader is probably most familiar with the curve of u' for Eq. (1), which corresponds to the rise of electric current in a series L-R circuit with fixed circuit elements,

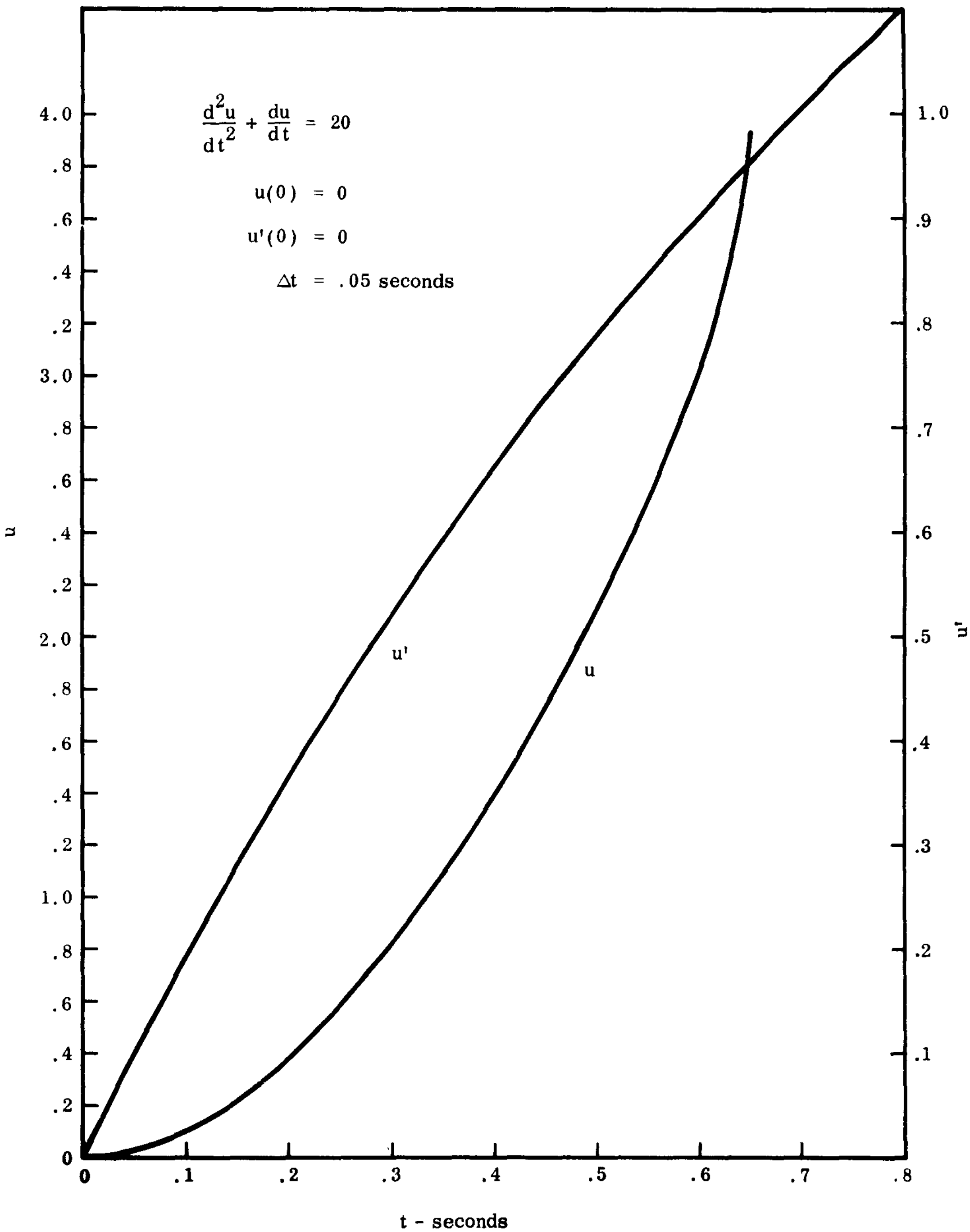


Fig. 5-1a Plot of $\frac{d^2u}{dt^2} + \frac{du}{dt} = 20$: u and u' vs. t

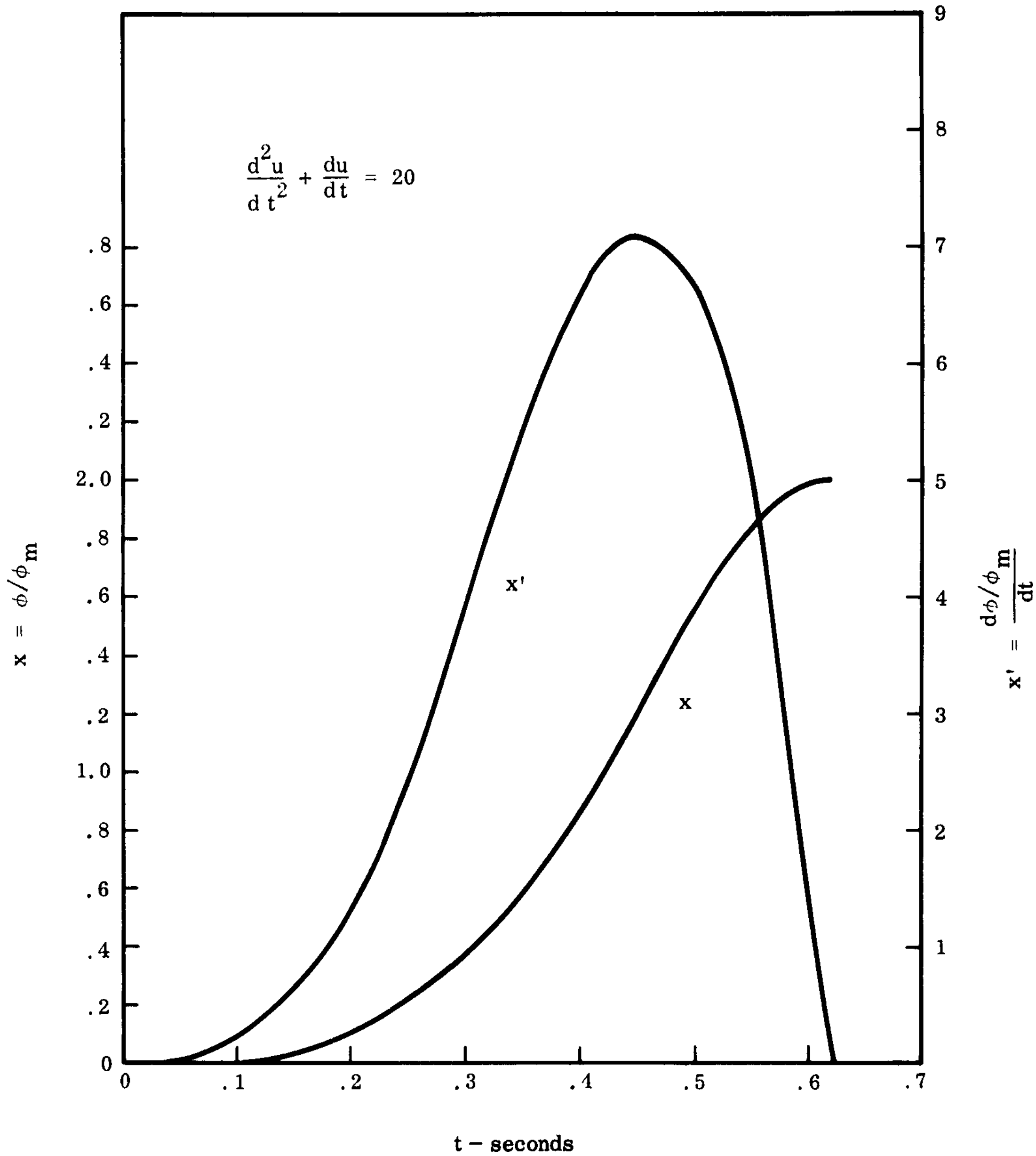


Fig. 5-1b Plot of $\frac{d^2u}{dt^2} + \frac{du}{dt} = 20$: x and x' vs. t

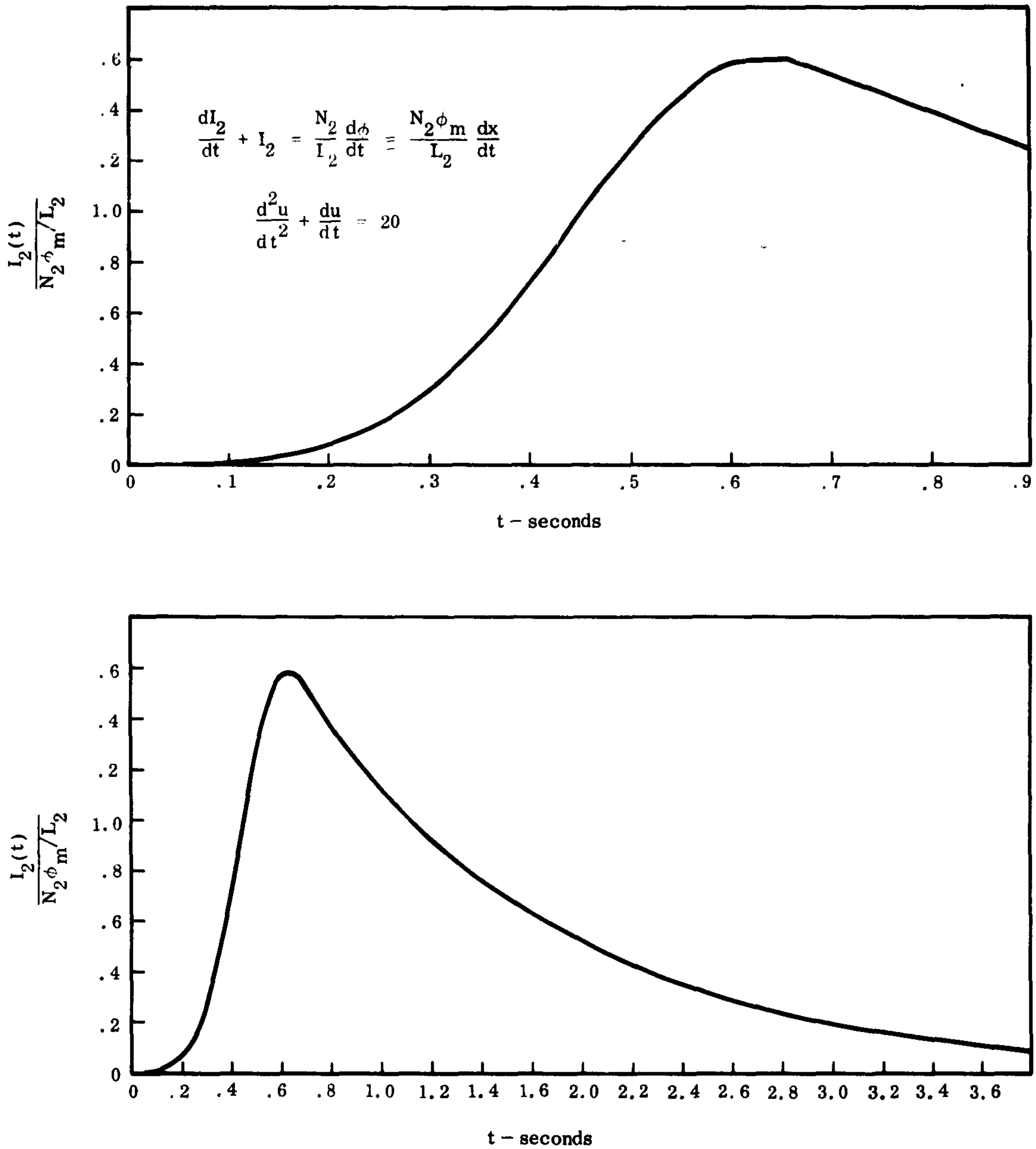


Fig. 5-1c Plot of $\frac{d^2u}{dt^2} + \frac{du}{dt} = 20$: Normalized Current vs. t

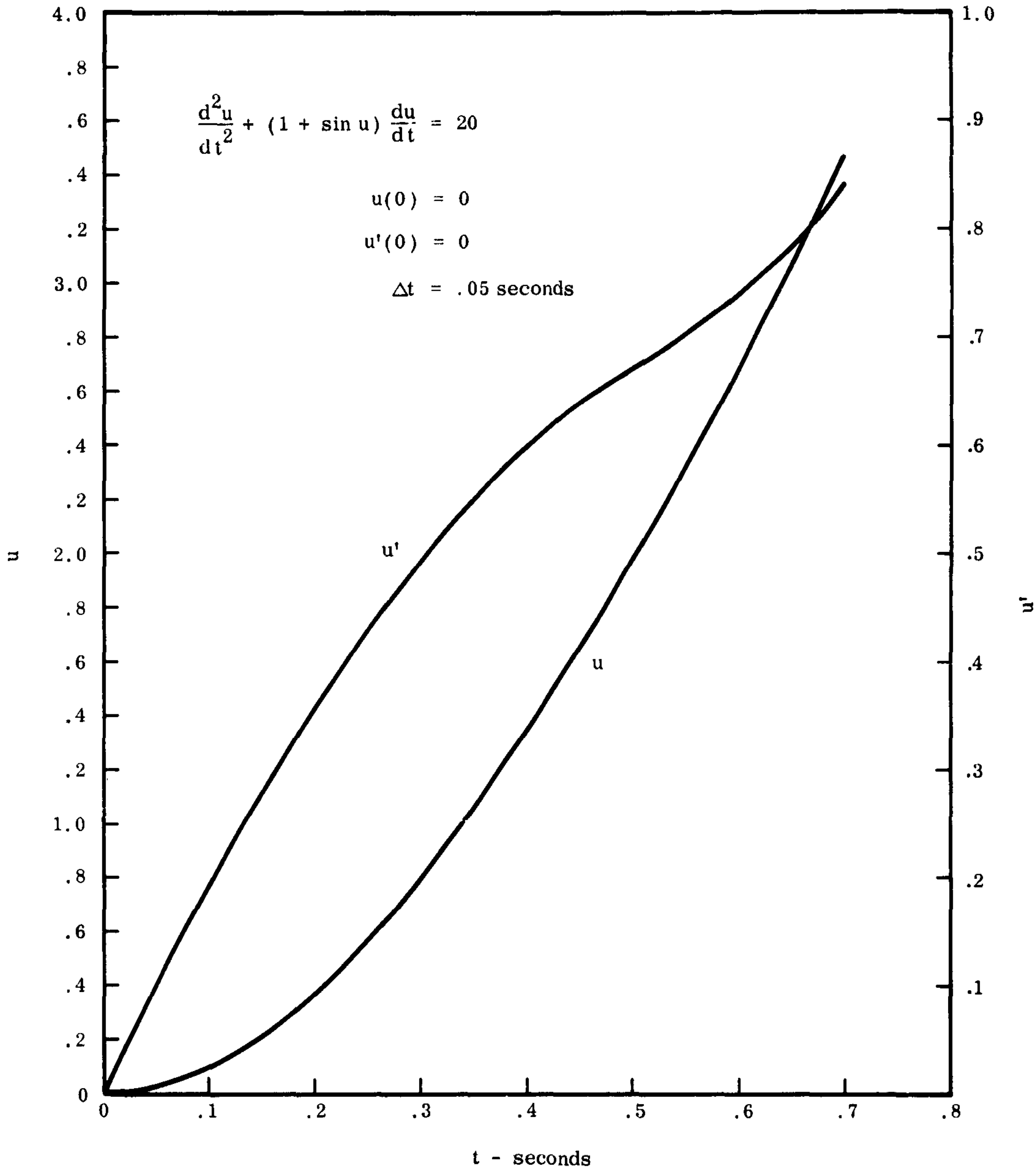


Fig. 5-2a Plot of $\frac{d^2u}{dt^2} + (1 + \sin u)\frac{du}{dt} = 20$: u and u' vs. t

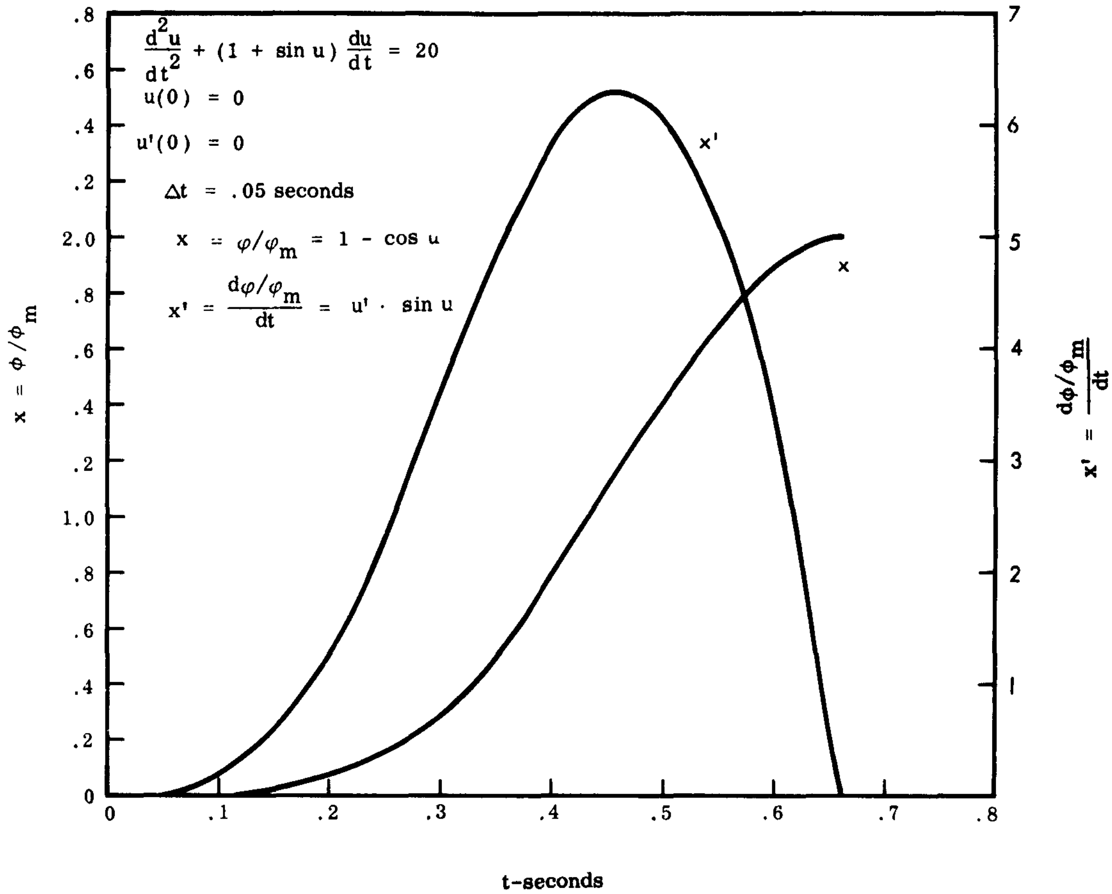
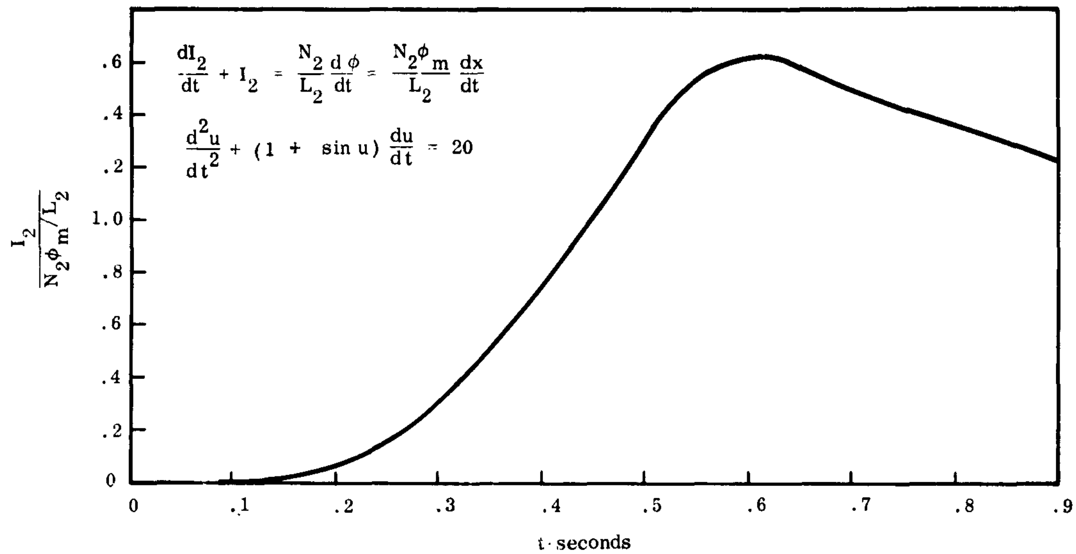


Fig. 5-2b Plot of $\frac{d^2u}{dt^2} + (1 + \sin u)\frac{du}{dt} = 20$: x and x' vs. t



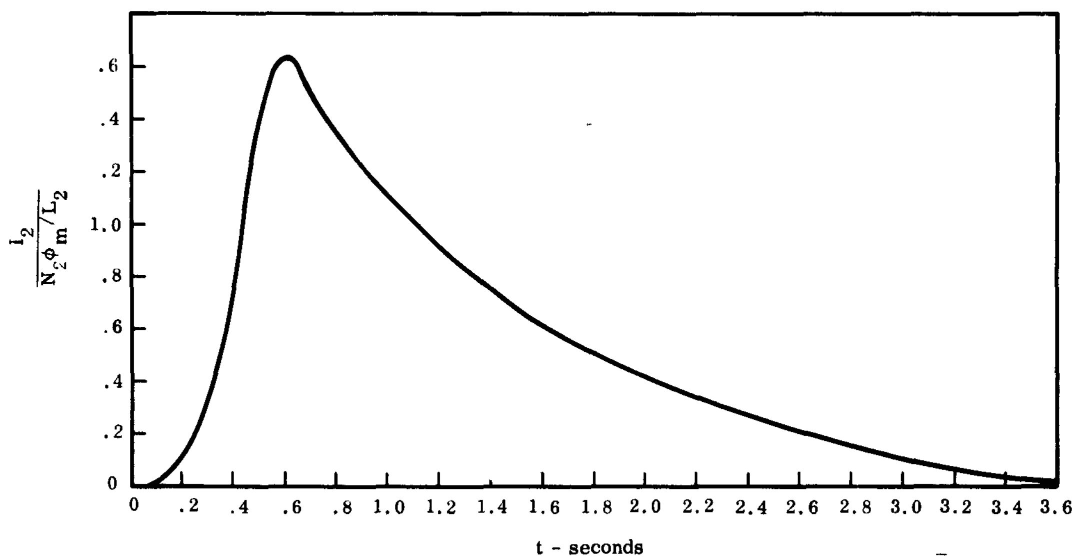


Fig. 5-2c Plot of $\frac{d^2u}{dt^2} + (1 + \sin u)\frac{du}{dt} = 20$: Normalized Current vs. t

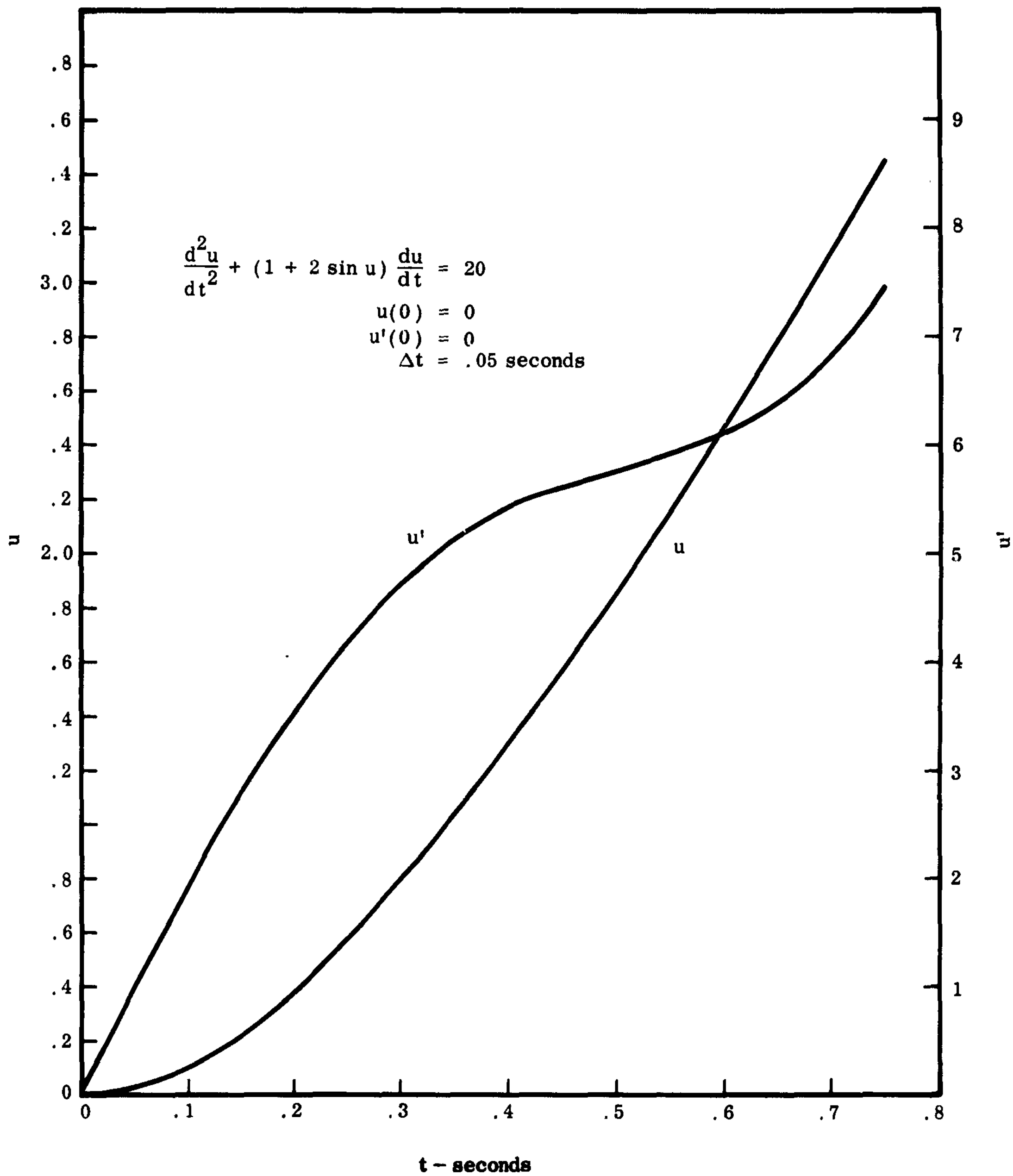


Fig. 5-3a Plot of $\frac{d^2u}{dt^2} + (1 + 2 \sin u) \frac{du}{dt} = 20$: u and u' vs. t

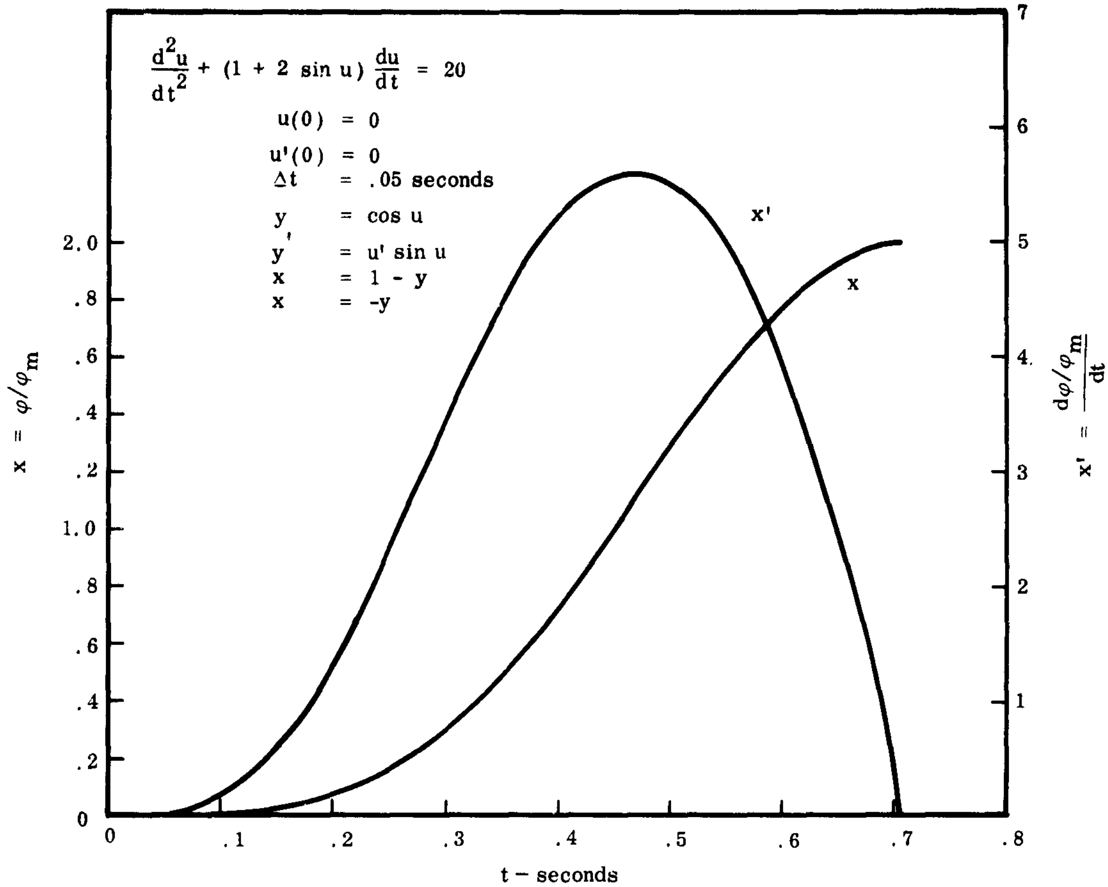
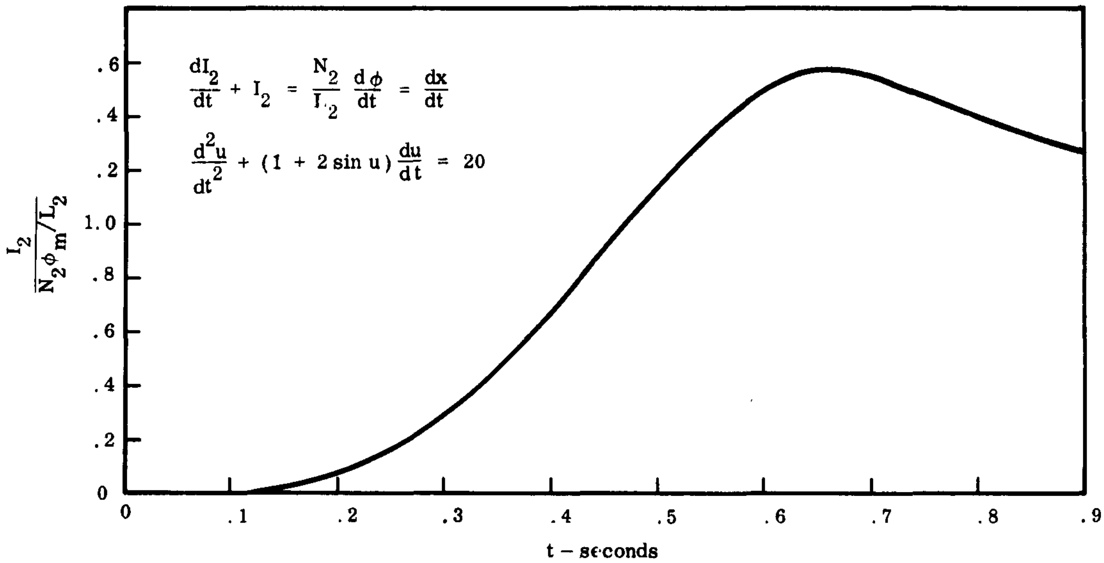


Fig. 5-3b Plot of $\frac{d^2u}{dt^2} + (1 + 2 \sin u) \frac{du}{dt} = 20$: x and x' vs. t



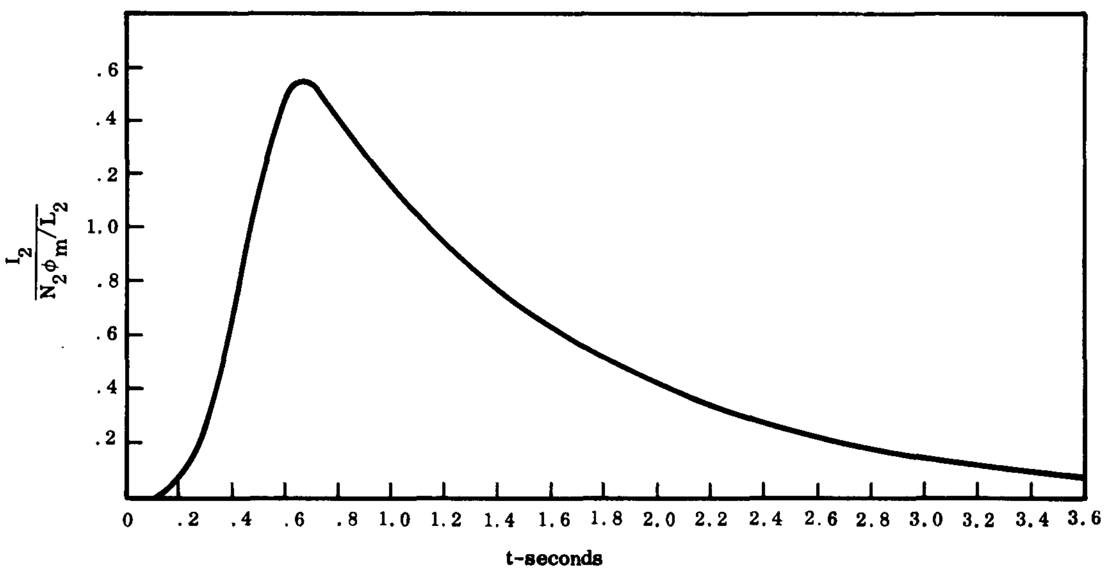


Fig. 5-3c Plot of $\frac{d^2u}{dt^2} + (1 + 2\sin u)\frac{du}{dt} = 20$: Normalized Current vs. t

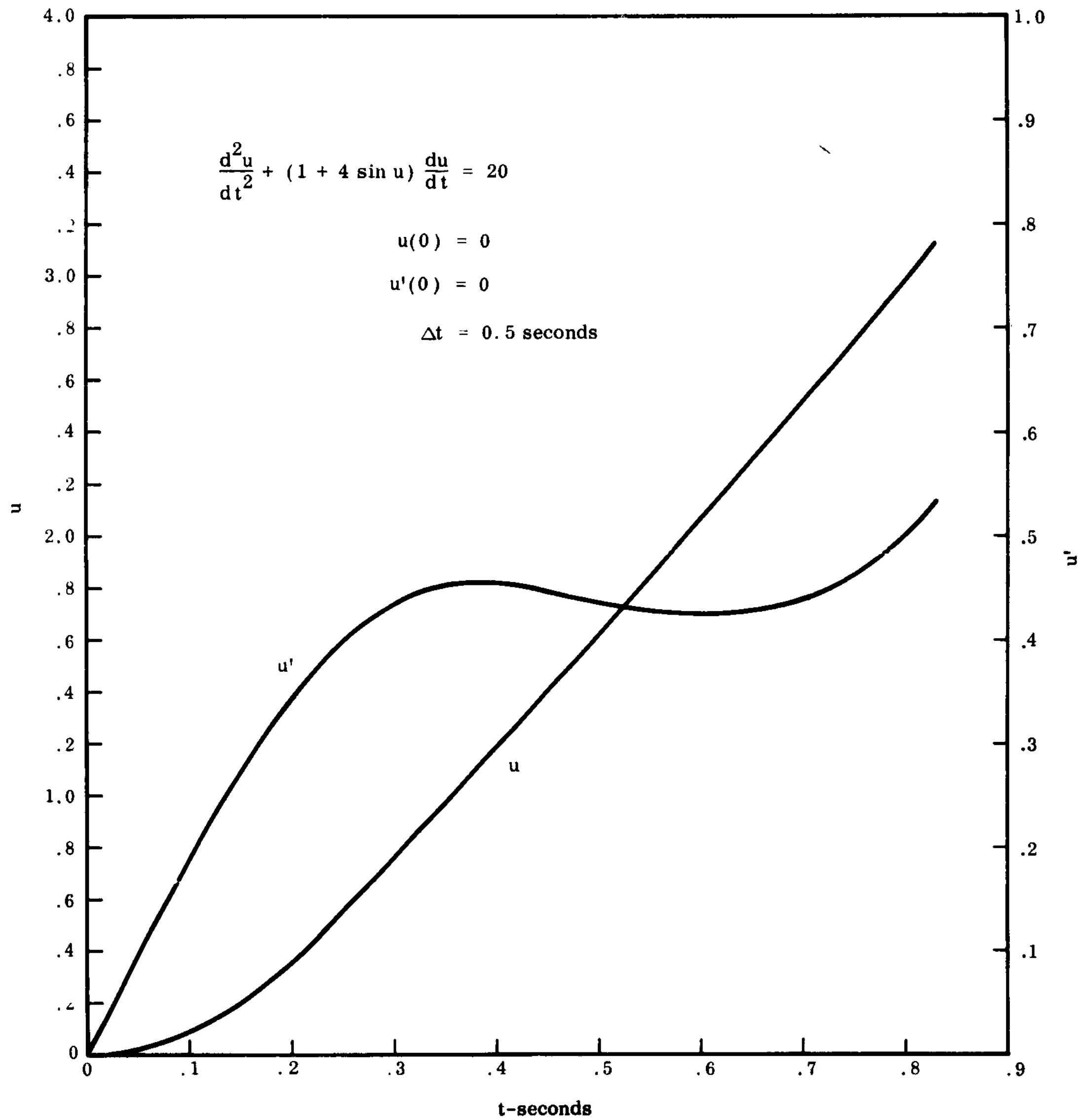


Fig. 5-4a Plot of $\frac{d^2u}{dt^2} + (1 + 4 \sin u) \frac{du}{dt} = 20$: u and u' vs. t

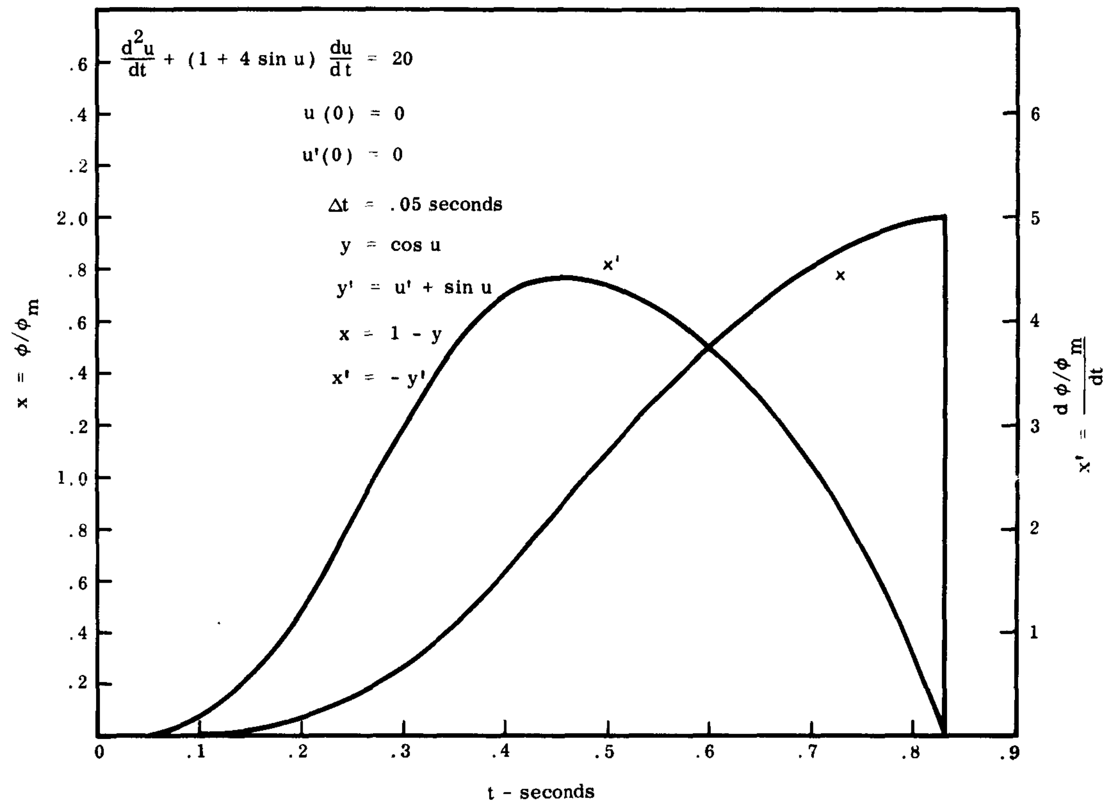
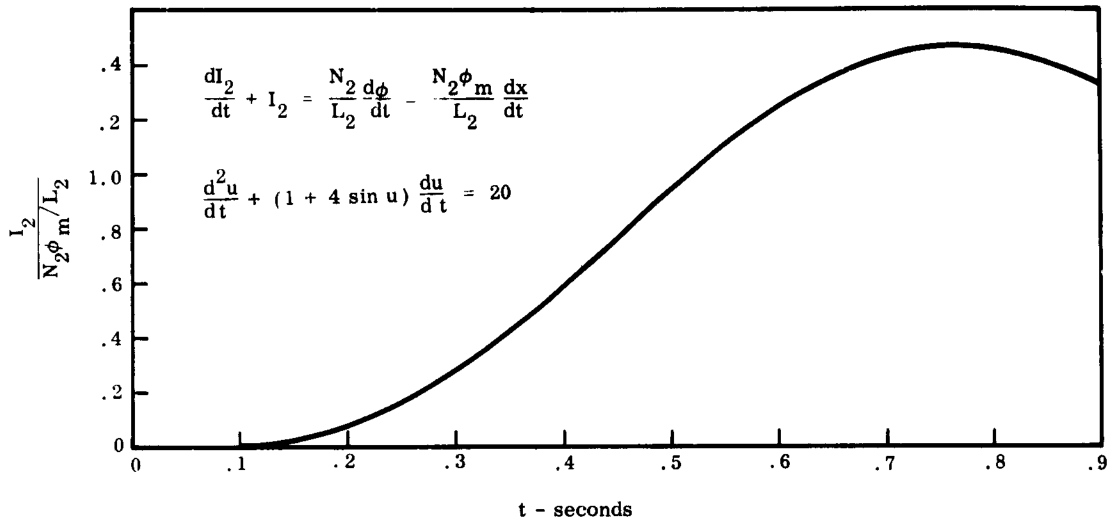


Fig. 5-4b Plot of $\frac{d^2u}{dt^2} + (1 + 4 \sin u) \frac{du}{dt} = 20$: x and x' vs. t



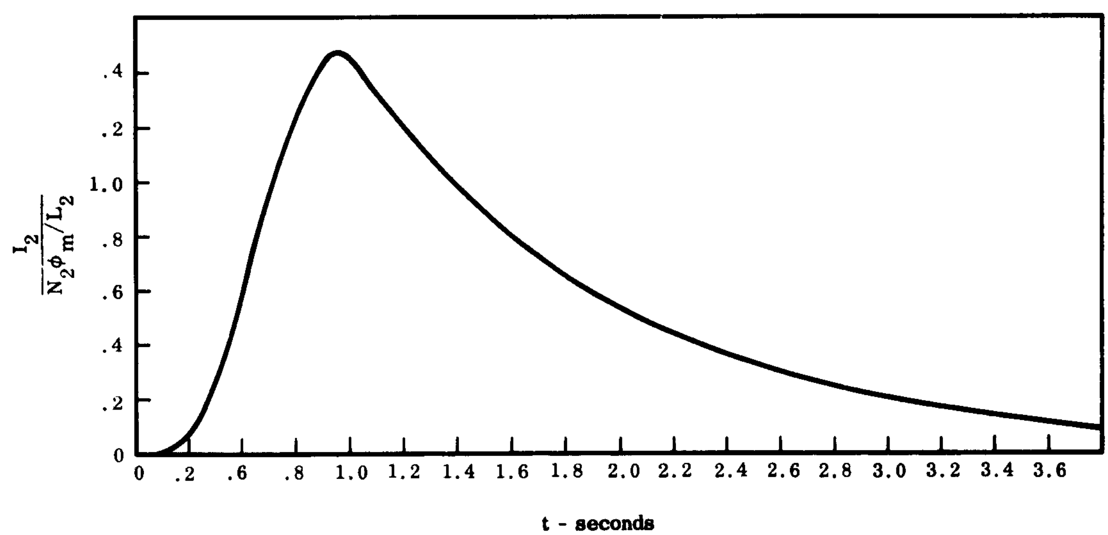


Fig. 5-4c Plot of $\frac{d^2u}{dt^2} + (1 + 4 \sin u)\frac{du}{dt} = 20$: Normalized Current vs. t

energized by a battery. In view of the analogy cited in Section 2 (see Fig. 2-4) the pertinent equation for the circuit charge is:

$$\frac{d^2q}{dt^2} + \frac{dq}{dt} = 20 \tag{5}$$

If we change the dependent variable from charge to current by means of the following substitution:

$$I = \frac{dq}{dt} \tag{6}$$

this becomes:

$$\frac{dI}{dt} + I = 20 \tag{7}$$

The solution is:

$$I = 20(1 - e^{-t}) \tag{8}$$

which represents an exponential rise of current to a final value of 20 amperes with a time constant of 1. The variable u' has the following identical form:

$$u' = 20(1 - e^{-t}) \tag{9}$$

of which only 8/10 of a time constant appears in Fig. 5-1a. Since the curve for u' is almost linear over this portion, the curve for u is almost parabolic. As the coefficient $N_2^2\, r/L_2$ of $\sin u$ increases (corresponding to increasing N_2 or r or both, L_2 remaining fixed), a pronounced wiggle appears in the curve for u'. This finally tends to make u' vary about a constant value over a portion of its range (as $u' = 4.375 \pm .125$ over the range $0.285 \leq t \leq .725$ in Fig. 5-4a) which causes the variation of u to change from parabolic to almost linear.

The output voltage, which is proportional to x', has the highest peak for the coefficient of $\sin u$ equal to zero, then decreasing from 7.1 through 6.3 and 5.6 to 4.45 for a coefficient of 4. Peak values of corresponding normalized output currents decrease from 1.64 through 1.59 and 1.56 to 1.48.

Section 6
CONCLUSIONS

Starting from Neeteson's approximation for the flux switching characteristic of a magnetic core, we have derived the differential equation describing the case when such a core is terminated in series inductance and resistance. By means of successive transformations, the equation was reduced to a form which could be solved by an iterative procedure. A family of solutions was computed and plotted.

To date no attempt has been made to check these curves against experimental results although verification is planned for the future. Since Neeteson's approximation is admittedly a compromise between exact fit to the flux characteristic and mathematical simplicity, only a fair agreement is to be expected.

It is believed that the differential equation and solutions are applicable to flux steering through multipath magnetic structures.

The mathematical methods employed are themselves of some interest.

Section 7
REFERENCES

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